Statistical inference-I

Statistical inference

The process of drawing inferences about population on the basis of sample information that contain the population is called statistical inference.

It has two types

i) Estimation of parameter ii) Testing of hypothesis

Estimation of parameter

Estimation is a procedure to estimate unknown value of population parameter on the basis of sample information.

It has two types

i) Point estimation ii) Interval estimation

Point estimation

The procedure to obtain a single unknown value of population parameter on the basis of sample information by using an estimator is called point estimation.

Point estimate/estimate

A single numerical value calculated from the sample data by using an estimator is called an estimate or point estimate. For example if we have 5, 10,15,20,25. Then $\bar{X} = \frac{\sum X}{n} = \frac{75}{5} = 15$.

Here 15 is a point estimate or estimate and $\frac{\sum X}{n}$ is estimator.

Interval estimation

The procedure to obtain an interval, on the belief that it will include the parameter θ with a known probability is called interval estimation.

Interval estimate

An interval, calculated on the belief that it will include the parameter θ with a known probability is called interval estimate. The confidence interval for θ with probability $(1-\alpha)$ is given by

$P(L \le \theta \le U) = (1-\alpha) \qquad For\ 0 \le \alpha \le 1$

Where "L" represents lower limit and "U" represent upper limit.

Define estimator

A formula/rule used to estimate the unknown value of population parameter on the basis of sample information is called estimate or estimator. Foe example $\bar{X} = \frac{\sum X}{n}$ here $\frac{\sum X}{n}$ is estimator.

What is meant by interval, confidence interval and confidence coefficient?

Interval: The range of values is called interval

Confidence interval: $(1-\alpha)$ or $100(1-\alpha)\%$ probability is associated with an interval that will contain the population parameter.

Confidence coefficient: An interval to which $100(1-\alpha)\%$ probability is associated that will contain the population parameter

How the width of confidence interval can be decreased?

The width of confidence interval can be decreased

i) By increasing the sample size

ii) Decreasing confidence coefficient (level of confidence)

If sample size is increased, what will be the change in confidence interval?

If sample size is increased, the width of confidence interval is decreased.

What will be the effect on a confidence interval if the level of confidence decreases?

The width of confidence interval decreases if the level of confidence decreases.

Differentiate between estimation, estimate and estimator?

Ans: In statistical inferences, **estimation** is a process; **estimator** is the formula and **estimates** the numerical value.

Differentiate between statistic, estimate and estimator?

Estimator is the formula and **estimates** the numerical value and statistic is a function of random variable being measured.

Why we construct confidence interval?

Since the point estimates may or may not be representative of the corresponding parameter, its reliability is doubtful. In order to increase reliability, we prefer interval estimation. Hence interval estimation is much more reliable then point estimation. The point estimate is used to obtain an interval estimate.

Define confidence limits.

The two endpoints of a confidence interval are called confidence limits.

Such as $P(L \leq \theta \leq U) = (1-\alpha) \qquad For\ 0 \leq \alpha \leq 1$

Where "L" represents lower limit and "U" represent upper limit.

What is error of estimation?

The distance between estimate and the estimated parameter is called error of estimation.

Define degree of freedom

Degree of freedom is the number of values that are free to vary after we have placed certain restrictions upon the data.

Properties of good estimator

The properties of good point estimator are given below

i) Unbiasedness ii) Consistency iii) Sufficiency iv) Efficiency

v) UMVUE (Uniform minimum variance unbiased estimator) vi) Completeness

vii) Invariance

If these properties satisfy then we say it is good point estimator

Property-I **Unbiased Estimator**

Unbiased estimator

An estimator $\hat{\theta}$ is said to be unbiased estimator of θ if and only if $E(\hat{\theta}) = \theta$ otherwise biased estimator $E(\hat{\theta}) \neq \theta$

i) If $E(\hat{\theta}) = \theta$ it is unbiased estimator
ii) $E(\hat{\theta}) \neq \theta$ it is biased
iii) $E(\hat{\theta}) > \theta$ it is positively biased
iv) $E(\hat{\theta}) < \theta$ it is negatively biased
This property is known as unbiasedness

Define unbiasedness

The property of an estimator being free from bias is called unbiasedness.

Differentiate between biased estimate and unbiased estimate?

An estimator is said to be unbiased if the mean of sampling distribution of the statistic is equal to its parameter. $E(\overline{X}) = \mu$ Otherwise it is biased $E(\overline{X}) \neq \mu$

Q. 1:

If 'T' is an unbiased estimator of $'\theta'$ then show that $'T^2'$ is a biased estimator of $'\theta^2'$.

Solution:

It is given that

$E(T) = \theta$

As we know that

$Var(T) \geq 0$

$E[T - E(T)]^2 \geq 0$

$E(T^2) - [E(T)]^2 \geq 0$

$E(T^2) - \theta^2 \geq 0$

$E(T^2) \geq \theta^2$

Hence $'T^2'$ is a biased estimator of $'\theta^2'$.

Q.2:

If 'T' is an unbiased estimator of $'\theta'$ then show that $'T^2'$ is a unbiased estimator of $'\theta^2'$.

Solution:

It is given that

$E(T) = \theta$

As we know that

$Var(T) = 0$

$$E[T-E(T)]^2 = 0$$

$$E(T^2)-[E(T)]^2 = 0$$

$$E(T^2)-\theta^2 = 0$$

$$E(T^2) = \theta^2$$

Hence $'T^2'$ is a unbiased estimator of $'\theta^2'$.

Q. 3: Statement: suppose that X_1, X_2, X_3, …,Xn be a random sample from a normal population distribution with zero mean and variance $'\sigma^2'$. Then show that $\frac{\sum X_i^2}{n}$ is an unbiased estimator of $'\sigma^2'$ with variance $\frac{2\sigma^4}{n}$.

Solution:

It gives that

$$X_i \approx N(0,\sigma^2)$$

We know that

$$X_i^2 \approx Z_i^2 \approx \sum Z_i^2 \approx \chi^2_{(n)} = \sum_{i=1}^{n}\left(\frac{X-\mu}{\sigma}\right)^2$$

Where"n" is degree of freedom And we have μ=0 then

$$\sum_{i=1}^{n}\frac{(X_i-0)^2}{\sigma^2} = \chi^2_{(n)}$$

$$\sum_{i=1}^{n}\frac{X_i^2}{\sigma^2} = \chi^2_{(n)}$$

$$\sum_{i=1}^{n}X_i^2 = \sigma^2\chi^2_{(n)}$$

Dividing both sides by 'n'

$$\frac{\sum_{i=1}^{n}X_i^2}{n} = \frac{\sigma^2\chi^2_{(n)}}{n} \qquad \text{(A)}$$

For mean

Applying expectation on both sides of equation (A)

$$E\left[\frac{\sum_{i=1}^{n}X_i^2}{n}\right] = E\left[\frac{\sigma^2}{n}\chi^2_{(n)}\right]$$

$$E\left[\frac{\sum_{i=1}^{n}X_i^2}{n}\right] = \frac{\sigma^2}{n}E\left[\chi^2_{(n)}\right]$$

$$E\left[\frac{\sum_{i=1}^{n} X_i^2}{n}\right] = \frac{\sigma^2}{n} n$$

$$E\left[\frac{\sum_{i=1}^{n} X_i^2}{n}\right] = \sigma^2$$

Hence $\frac{\sum X_i^2}{n}$ is an unbiased estimator of σ^2.

FOR VARIANCE

Applying variance on both sides of equation (A)

$$\operatorname{var}\left[\frac{\sum_{i=1}^{n} X_i^2}{n}\right] = \operatorname{var}\left[\frac{\sigma^2}{n}\chi^2_{(n)}\right]$$

$$E\left[\frac{\sum_{i=1}^{n} X_i^2}{n}\right] = \frac{\sigma^4}{n^2}\operatorname{var}\left[\chi^2_{(n)}\right]$$

$$E\left[\frac{\sum_{i=1}^{n} X_i^2}{n}\right] = \frac{\sigma^4}{n^2}2n$$

$$E\left[\frac{\sum_{i=1}^{n} X_i^2}{n}\right] = \frac{2\sigma^4}{n}$$

Hence $\frac{\sum X_i^2}{n}$ is an unbiased estimator of σ^2 with variance $\frac{2\sigma^4}{n}$.

Q.4:

Show that sample mean $\overline{X}$ is an unbiased estimator of Population mean μ and $S^2 = \frac{\sum (x-\bar{x})^2}{n}$ is biased Estimator of populationVariance σ^2 but $S^2 = \frac{\sum (x-\bar{x})^2}{n}$ is an unbiased estimator of population variance σ^2.

Solution:

a)

Let by definition

$$\overline{X} = \frac{\sum X_i}{n}$$

Applying expectation on both sides

$$E(\overline{X}) = \frac{E[X_1 + X_2 + ...+ X_n]}{n}$$

$$E(\overline{X}) = \frac{E(X_1) + E(X_2) + ...+ E(X_n)}{n}$$

$$E(\overline{X}) = \frac{\mu + \mu + ...+ \mu}{n} \qquad \text{Where } E(X) = \mu$$

$$E(\overline{X}) = \mu$$

Hence $\overline{X}$ is an unbiased estimator μ .

b) As we know that

$$S^2 = \frac{\sum (X - \overline{X})^2}{n}$$

$$nS^2 = \sum (X - \overline{X})$$

Adding and subtracting 'μ' inside bracket

$$nS^2 = \sum \left(X_i - \mu + \mu - \overline{X}\right)^2$$

$$nS^2 = \sum^{n} \left[(X_i - \mu) - (\overline{X} - \mu)\right]^2$$

$$nS^2 = \sum^{n} \left[(X_i - \mu)^2 + (\overline{X} - \mu)^2 - 2(X_i - \mu)(\overline{X} - \mu)\right]$$

$$nS^2 = \sum^{n} \left(X_i - \mu\right)^2 + \sum^{n} (\overline{X} - \mu)^2 - 2(\overline{X} - \mu)\sum^{n} (X_i - \mu)$$

$$\text{Where } \sum^{n} (X_i - \mu) = \sum^{n} X_i - n\mu = n\overline{X} - n\mu = n(\overline{X} - \mu)$$

$$nS^2 = \sum^{n} \left(X_i - \mu\right)^2 + n(\overline{X} - \mu)^2 - 2(\overline{X} - \mu)n(\overline{X} - \mu)$$

$$nS^2 = \sum^{n} \left(X_i - \mu\right)^2 + n(\overline{X} - \mu)^2 - 2n(\overline{X} - \mu)^2$$

$$nS^2 = \sum^{n} \left(X_i - \mu\right)^2 - n(\overline{X} - \mu)^2$$

Applying expectation on both sides

$$nE(S^2) = \sum^{n} E\left(X_i - \mu\right)^2 - nE(\overline{X} - \mu)^2$$

$$nE(S^2) = n\sigma^2 - n\,\mathrm{var}(\overline{X})$$

Where $E(X_i-\mu)^2=\sigma^2$

$$nE(S^2)=n\sigma^2-n\frac{\sigma^2}{n}$$

$$nE(S^2)=\sigma^2(n-1)$$

$$E(S^2)=\frac{\sigma^2}{n}(n-1)$$

Hence S^2 is a biased estimator of σ^2.

(C)

As we know that

$$s^2=\frac{\sum(X-\overline{X})^2}{n-1}$$

$$(n-1)s^2=\sum(X-\overline{X})$$

Adding and subtracting 'μ' inside bracket

$$(n-1)s^2=\sum\left(X_i-\mu+\mu-\overline{X}\right)^2$$

$$(n-1)s^2=\sum^n\left[(X_i-\mu)-(\overline{X}-\mu)\right]^2$$

$$(n-1)s^2=\sum^n\left[(X_i-\mu)^2+(\overline{X}-\mu)^2-2(X_i-\mu)(\overline{X}-\mu)\right]$$

$$(n-1)s^2=\sum^n\left(X_i-\mu\right)^2+\sum^n(\overline{X}-\mu)^2-2(\overline{X}-\mu)\sum^n(X_i-\mu)$$

Where $\sum^n(X_i-\mu)=\sum^n X_i-n\mu=n\overline{X}-n\mu=n(\overline{X}-\mu)$

$$(n-1)s^2=\sum^n\left(X_i-\mu\right)^2+n(\overline{X}-\mu)^2-2(\overline{X}-\mu)n(\overline{X}-\mu)$$

$$(n-1)s^2=\sum^n\left(X_i-\mu\right)^2+n(\overline{X}-\mu)^2-2n(\overline{X}-\mu)^2$$

$$(n-1)s^2=\sum^n\left(X_i-\mu\right)^2-n(\overline{X}-\mu)^2$$

Applying expectation on both sides

$$(n-1)E(s^2)=\sum^n E\left(X_i-\mu\right)^2-nE(\overline{X}-\mu)^2$$

$$(n-1)E(s^2)=n\sigma^2-n\,\mathrm{var}(\overline{X})$$

Where $E(X_i-\mu)^2=\sigma^2$

$$(n-1)E(s^2) = n\sigma^2 - n\frac{\sigma^2}{n}$$

$$(n-1)E(s^2) = \sigma^2(n-1)$$

$$E(s^2) = \sigma^2$$

Hence s^2 is a unbiased estimator of σ^2.

Q.5:

Let $X_1, X_2, X_3 ... X_n$ be a random sample of size 3 from a normal population with mean μ and (σ^2) T_1, T_2, T_3 are the estimators defined as

$$T_1 = X_1 + X_2 - X_3$$

$$T_2 = 2X_1 + 3X_2 - 4X_3$$

$$T_3 = \frac{\lambda X_1 + X_2 + X_3}{3}$$

Show that

a) T_1 and T_2 are unbiased estimator of μ
b) Find the value of λ such that 'T_3' is an unbiased estimator of μ
c) Which one are the best estimators.

Solution:

a) Let

$$T_1 = X_1 + X_2 - X_3$$

Applying expectation on both side

$$E(T_1) = E(X_1 + X_2 - X_3)$$

$$E(T_1) = E(X_1) + E(X_2) - E(X_3)$$

$$E(T_1) = \mu + \mu - \mu$$

$$E(T_1) = \mu$$

Hence $T_1 = X_1 + X_2 - X_3$ is an unbiased estimator of μ

(ii)

$$T_2 = 2X_1 + 3X_2 - 4X_3$$

Applying expectation on both sides

$$E(T_2) = 2E(X_1) + 3E(X_2) - 4E(X_3)$$

$$E(T_2) = 2\mu + 3\mu - 4\mu$$

$$E(T_2) = \mu$$

Hence $T_2 = 2X_1 + 3X_2 - 4X_3$ is an unbiased estimator of μ

b)

To find value of λ

As

$$T_3 = \frac{\lambda X_1 + X_2 + X_3}{3}$$

$$3T_3 = \lambda X_1 + X_2 + X_3$$

Applying expectation on both sides

$$3E(T_3) = \lambda E(X_1) + E(X_2) + E(X_3)$$

$$3E(T_3) = \lambda E(X_1) + E(X_2) + E(X_3)$$

$$3\mu = \lambda\mu + \mu + \mu$$

$$3\mu = \lambda\mu + 2\mu$$

$$\mu = \lambda\mu$$

$$\lambda = 1$$

So $\quad T_3 = \dfrac{X_1 + X_2 + X_3}{3}$

c) Variances

$$T_1 = X_1 + X_2 - X_3$$

Appling variance on both sides

$$Var(T_1) = Var(X_1) + Var(X_2) - Var(X_3)$$

$$Var(T_1) = \sigma^2 + \sigma^2 + \sigma^2$$

$$Var(T_1) = 3\sigma^2$$

Similarly

ii) $T_2 = 2X_1 + 3X_2 - 4X_3$

Appling variance on both sides

$$Var(T_2) = 4Var(X_1) + 9Var(X_2) - 16Var(X_3)$$

$$Var(T_2) = 4\sigma^2 + 9\sigma^2 + 16\sigma^2$$

$$Var(T_2) = 29\sigma^2$$

iii)

$$T_3 = \frac{X_1 + X_2 + X_3}{3}$$

Applying variance on both sides

$$Var(T_3) = \frac{Var(X_1) + Var(X_2) + Var(X_3)}{9}$$

$$Var(T_3) = \frac{\sigma^2 + \sigma^2 + \sigma^2}{9}$$

$$Var(T_3) = \frac{3\sigma^2}{9}$$

$$Var(T_3) = \frac{\sigma^2}{3}$$

The relation exist

$Var(T_2) > Var(T_1) > Var(T_3)$

So T_3 are the best estimators

Q.6:

Let $X_1, X_2 ... X_n$ be a random sample size 'n' observation and $Y_1, Y_2 ... Y_n$ be a random sample of size 'm' observation then show that $\overline{X}_{p=} \frac{n\overline{X} + m\overline{Y}}{n+m}$ and $S_p^2 = \frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2}$

Are unbiased estimators of μ and σ^2 respectively.

Solution:

It is given that

$$\overline{X}_p = \frac{n\overline{X} + m\overline{Y}}{n+m}$$

Applying expectation on both sides

$$E(\overline{X}_p) = E\left[\frac{n\overline{X} + m\overline{Y}}{n+m}\right]$$

$$\overline{X} = \frac{\sum X}{n}, S_x^2 = \frac{\sum(X - \overline{X})^2}{n-1} \quad \overline{Y} = \frac{\sum Y}{m}, S_y^2 = \frac{\sum(Y - \overline{Y})^2}{m-1}$$

$$\overline{X}_p = \frac{nE\overline{X} + mE\overline{Y}}{n+m}$$

$$E(\overline{X}) = \mu \quad and \quad E(\overline{Y}) = \mu$$

$$\overline{X}_p = \frac{n\mu + m\mu}{n+m} = \frac{\mu(n+m)}{(n+m)}$$

$E(\overline{X}_p) = \mu$ Hence $\overline{X}_p = \frac{n\overline{X} + m\overline{Y}}{n+m}$ is an unbiased estimator of μ.

Now

$$S_p^2 = \frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2}$$

Appling expectation on both sides

$$E(S_p^2) = E\left[\frac{(n-1)S_x^2 + (m-1)S_y^2}{n+m-2}\right]$$

$$E(S_p^2) = \left[\frac{(n-1)E(S_x^2) + (m-1)E(S_y^2)}{n+m-2}\right]$$

$$E(S_p^2) = \frac{(n-1)E(S_x^2) + (m-1)E(S_y^2)}{n+m-2}$$

$$E(S_p^2) = \frac{(n-1)E(S_x^2) + (m-1)E(S_y^2)}{n+m-2} = \frac{(n-1)\sigma^2 + (m-1)\sigma^2}{n+m-2}$$

$$E(S_p^2) = \frac{\sigma^2(n-1+m-1)}{n+m-2}$$

$$E(S_p^2) = \sigma^2$$

Hence $S_p^2 = \dfrac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2}$ it is an unbiased estimator of σ^2.

Q.7

(a) If 'x' is a passion random variable then show that $E(e^{-kx}) = e^{-m(1-e^{-k})}$

(b)Show that if $\overline{X}$ the mean of 'n' passion variates an independent random 'm' then show that $e^{-\overline{X}}$ is biased estimator of e^{-m}.

Solution:

(a)

As $X \approx P(x;m)$

$$P(X=x) = \frac{e^{-m}m^x}{x!} \qquad X = 0,1,2,3,......,\infty$$

As

$$E(e^{-kx}) = \sum_{x=0}^{\infty} e^{-kx}P(X)$$

$$E(e^{-kx}) = \sum_{x=0}^{\infty} e^{-kx}\frac{e^{-m}m^x}{x!}$$

$$E(e^{-kx}) = e^{-m}\sum_{x=0}^{\infty} \frac{e^{-kx}m^x}{x!}$$

$$E(e^{-kx}) = e^{-m}\left[\frac{(e^{-k}m)^x}{x!}\right]$$

$$E(e^{-kx}) = e^{-m}\left[\frac{(e^{-k}m)^0}{0!} + \frac{(e^{-k}m)^1}{1!} + \frac{(e^{-k}m)^2}{2!} + ...\right]$$

$$E(e^{-kx}) = e^{-m}\left[1 + \frac{(e^{-k}m)}{1} + \frac{(e^{-k}m)^2}{2} + ...\right]$$

Therefore $e^\theta = \left[1 + \dfrac{\theta}{1!} + \dfrac{\theta}{2!} + ...\right]$

$$E(e^{-kx}) = e^{-m}e^{me^{-k}}$$

$$E(e^{-kx}) = e^{-m+me^{-k}}$$

$$E(e^{-kx}) = e^{-m(1-e^{-k})}$$

Hence proved

(b)

Now we consider

$$E(e^{-\overline{X}}) = E(e^{-\frac{\sum X}{n}})$$

$$E(e^{-\overline{X}}) = E[e^{-\frac{X_1}{n}-\frac{X_2}{n}-.........-\frac{X_n}{n}}]$$

$$E(e^{-\overline{X}}) = E[e^{-\frac{X_1}{n}}e^{-\frac{X_2}{n}}..........e^{-\frac{X_n}{n}}]$$

As, X's are independent

$$E(e^{-\overline{X}}) = E[(e^{-\frac{X_1}{n}})E(e^{-\frac{X_2}{n}})..........E(e^{-\frac{X_n}{n}})$$

$$E(e^{-\overline{X}}) = \prod_{i=1}^{n} E[(e^{-\frac{X_i}{n}})$$

AS $\frac{1}{n} = k$

$$E(e^{-\overline{X}}) = \prod_{i=1}^{n} E[(e^{-\frac{X_i}{n}})$$

AS

$$E[(e^{-\frac{1}{n}X_i}) = E[e^{-km}] = e^{-m(1-e^{-k})}$$

$$E(e^{-\overline{X}}) = \prod_{i=1}^{n} e^{-m}(1-e^{-k})$$

$$E(e^{-\overline{X}}) = e^{-nm(1-e^{-k})}$$

$$E(e^{-\overline{X}}) \neq e^{-m}$$

Hence $e^{-\overline{X}}$ is biased estimator of e^{-m}

Q.8:

Let $X_1, X_2 X_3, \ldots X_n$ be a random sample from a Bernoulli variable "X" taking the value "1" with probability "θ" and value "0" with probability "1-θ". Show that $\frac{T(T-1)}{n(n-1)}$ is an unbiased estimator of θ^2. Where $T = \sum X_i$

Solution:

$$As \quad X \approx bernulli \quad (\theta)$$

$$p(X = x) = \binom{1}{x} \theta^x (1-\theta)^{1-x} \qquad X = 0,1,2,3....$$

$$\hat{\theta} = \frac{T^2 - T}{n(n-1)}$$

Taking expectation on both side

$$E(\hat{\theta}) = E[\frac{T^2 - T}{n(n-1)}]$$

$$E(\hat{\theta}) = \frac{E(T^2) - E(T)}{n(n-1)} \qquad \text{(A)}$$

AS

$$T = \sum X_i$$

$$E(T) = \sum E(X_i)$$

$$E(T) = n\,E(X_i)$$

$E(T) = n\theta$ In case of Bernoulli distribution $E(X_i) = \theta$

Now

$$E(T^2) = E[\sum X]^2$$

$$E(T^2) = E[\sum X_i^2 + \sum_{i}^{n} \sum_{\neq j}^{n-1} X_i X_j]$$

$$E(T^2) = nE(X_i)^2 + n(n-1)E(X_i)E(X_j)$$

$$E(T^2) = nE(X_i)^2 + n(n-1)\theta.\theta$$

In case of Bernoulli distribution $E(X_i)^2 = \theta$

$E(T^2) = n\theta + n(n-1)\theta^2$ Put the values in equation (A)

$$E(\hat{\theta}) = \frac{n\theta + n(n-1)\theta^2 - n\theta}{n(n-1)}$$

$$E(\hat{\theta}) = \frac{n(n-1)\theta^2}{n(n-1)}$$

$$E(\hat{\theta}) = \theta^2$$

Hence $\frac{T(T-1)}{n(n-1)}$ *is an unbised estimator of* θ^2

Q. 9:

Given a random sample of size ‘n’ from a population with mean ‘μ’ and finite variance ‘σ^2’.show that $\frac{1}{n}\sum(x-\mu)^2$ is an unbiased estimation of ‘σ^2.

Solution :

Let $T = \frac{1}{n}\sum (x-\mu)^2$

Applying expectation on both side

$$E(T) = \frac{1}{n}E[(x-\mu)^2]$$

$$E(T) = \frac{1}{n}E[(x_1-\mu)^2 + (x_2-\mu)^2 + (x_3-\mu)^2 + ...+(x_n-\mu)^2]$$

$$E(T) = \frac{1}{n}[E(x_1-\mu)^2 + E(x_2-\mu)^2 + E(x_3-\mu)^2 + ...+E(x_n-\mu)^2]$$

Therefore $E(x-\mu) = \sigma^2$

$$E(T) = \frac{1}{n}[\sigma^2 + \sigma^2 + \sigma^2 + ...+\sigma^2]$$

$$E(T) = n\frac{\sigma^2}{n} = \sigma^2$$

Hence $\frac{1}{n}\sum (x-\mu)^2$ is an unbiased estimation of ‘ σ^2 .

Q.10:

If $\overline{X}_1$ & $\overline{X}_2$ are the mean of independent variable of size n_1 & n_2 from a normal population with mean μ and variance σ^2. Show that $\theta \overline{X}_1 + (1-\theta)\overline{X}_2$ is an unbiased estimator of μ.

Solution:

Let $T = \theta \overline{X}_1 + (1-\theta)\overline{X}_2$

Applying expectation on both sides.

$E(T) = E[\theta \overline{X}_1 + (1-\theta)\overline{X}_2]$

$E(T) = \theta E(\overline{X}_1) + (1-\theta)E(\overline{X}_2)$

$E(T) = \theta\mu + (1-\theta)\mu$ Therefor $E(\overline{x}_1) = \mu$

$E(T) = \theta\mu + \mu - \theta\mu$

$E(T) = \mu$

Hence $\theta \overline{X}_1 + (1-\theta)\overline{X}_2$ is an unbiased estimator of μ.

Q.11:

Given that $f(x,\theta) = \frac{X^{\theta-1}e^{-X}}{\Gamma\theta}$ $X \geq 0$

a) Find the value of ‘c’ such that dx=cx will be an unbiased estimator of θ.
b) Determine for dx=c x^2 *where dx is an unbiased estimator of* θ.
Solution:
Let

$$dx = CX$$

Applying expectation on both sides

$E(dx) = CE(X)$

$E(X) = \int_{-\infty}^{\infty} xf(x)dx$

$E(X) = \int_{-\infty}^{\infty} xf(x)dx$

As x ≈ Gamma(θ)

f(x) = $\frac{1}{\Gamma\theta} x^{\theta-1} e^{-x}$

$E(X) = \int_{0}^{\infty} XX^{\theta-1} \frac{1}{\Gamma\theta} e^{-x} dx$

$E(X) = \frac{1}{\Gamma\theta} \int_{0}^{\infty} X^{\theta+1-1} e^{-x} dx$ (A)

As we know that gamma function is:

$\Gamma a b^{a} = \int_{0}^{\infty} x^{a-1} e^{-x/b} dx$ **(B)**

Comparing (A) & (B) and we get

$a = \theta + 1 \quad and \quad b = 1$

$E(X) = \frac{1}{\Gamma\theta} \Gamma\theta + 1 = \theta$

$E(dx) = CE(X)$ As $E(dx) = \theta$

$\theta = C\theta$

$C = 1$

b)

Let

$dx = cX^{2}$

$dx = cX^{2}$

Apply expectation on both sides

$E(dx) = cE(X)^{2}$

$E(dx) = c\int_{-\infty}^{\infty} x^{2} f(x)dx$

As X→G($\boldsymbol{\theta}$)

$$E(dx) = c\int_0^\infty x^2 \frac{x^{\theta-1}e^{-x}}{\Gamma\theta}dx$$

$$E(dx) = c\int_0^\infty \frac{x^{\theta+2-1}e^{-x}}{\Gamma\theta}dx \qquad \text{(A)}$$

As we know that gamma function

$$\Gamma\alpha\,\beta^\alpha = \int_0^\infty x^{\alpha-1}e^{-\frac{x}{\beta}}dx \qquad \text{(B)}$$

Comparing (A) & (B)

$\alpha = \theta + 2$ and $\beta = 1$

Put in eq(A)

$$E(dx) = c\frac{\Gamma\theta+2}{\Gamma\theta} = C\frac{(\theta+1)\theta\Gamma\theta}{\Gamma\theta} = C(\theta+1)\theta$$

$$\boldsymbol{\theta = C(\theta+1)\theta}$$

$$\boldsymbol{C = \frac{1}{(\theta+1)}}$$

Q.12

Let $X_{(1)}^3 < X_{(2)}^3 < X_{(3)}^3$ be the order statistic of size '3' from the uniform distribution having p. d .f $f(x) = \frac{1}{\theta}$ $0 < X < \theta$. Then show that $4X_{(1)}^3, 2X_{(2)}^3, \frac{4}{3}X_{(3)}^3$ are unbiased estimator of θ . Also find their variance.

Solution:

The p .d .f of ith order statistics

$$g(y_{(i)}) = \frac{n!}{(\text{i}-1)!\,(\text{n}-\text{i})!}\left[F(y_i)\right]^{i-1}\left[1-F(y_{i)}\right]^{n-i} f(y_i)$$

(i) $4X_{(1)}^3$

Put i=1 , n=3 , $y_i = x_i$

$$g(y_{(i)}) = \frac{n!}{(\text{i}-1)!\,(\text{n}-\text{i})!}\left[F(y_i)\right]^{i-1}\left[1-F(y_{i)}\right]^{n-i} f(y_i)$$

$$g(X_{(1)}) = \frac{3!}{0!\,2!}\left[F(X_{(1)})\right]^0\left[1-F(X_{(1)}\right]^{3-1} f(X_{(1)})$$

$$g(X_{(1)}) = 3\left[1-F(X_{(1)}\right]^2 f(X_{(1)}) \qquad \text{(A)}$$

As

$$f(x)=\frac{1}{\theta}\quad 0<X<\theta$$

$$F(x)=\int_0^X f(X)d(X)==\int_0^X \frac{1}{\theta}1d(X)$$

$$F(x)=\frac{1}{\theta}X\Big|_0^x$$

$$F(x)=\frac{X}{\theta}$$

$$F(x_1)=\frac{X_{(1)}}{\theta}$$

$$f(x_1)=\frac{1}{\theta}$$

Now Esq. (B) becomes

$$g(X_{(i)})=3\left[1-\frac{X(1)}{\theta}\right]^2\frac{1}{\theta}$$

$$g(X_{(i)})=\frac{3}{\theta}\left[1-\frac{X(1)}{\theta}\right]^2$$

Now

$$E\left[4X_{(1)}\right]=\int 4X_{(1)}g(X_{(1)})dX_{(1)}$$

$$E\left[4X_{(1)}\right]=4\int_0^\theta \frac{3}{\theta}\left(1-\frac{X_{(1)}}{\theta}\right)^2 X_{(1)}dX_{(1)}$$

$$\beta(a,b)=\int_0^1 W^{a-1}(1-w)^{b-1}dW \qquad \text{(B)}$$

Put

$$W=\frac{X_{(1)}}{\theta} \qquad X_{(1)}\longrightarrow 0 \text{ Then } \quad W\longrightarrow 0$$

$$W\theta=X_{(1)}\quad X_{(1)}\longrightarrow\theta \qquad \text{Then} \quad W\longrightarrow 1$$

$$dX_{(1)}=\theta dW$$

$$E\left[4X_{(1)}\right]=\frac{12}{\theta}\int_0^1 W\theta\,(1-W)^2\theta dW$$

$$E\left[4X_{(1)}\right]=12\theta\int_0^1 W^{2-1}\,(1-W)^{3-1}dW \qquad ©$$

As we know that beta function of 1st kind

$$E\left[4X_{(1)}\right]=\frac{12}{\theta}\int_0^\theta X_{(1)}\left(1-\frac{X_{(1)}}{\theta}\right)^2 dX_{(1)} \qquad \text{(D)}$$

Comparing (C) & (D)

a=2 , b= 3

$$E\left[4X_{(1)}\right]=12\theta\beta(2,3)$$

$$E\left[4X_{(1)}\right]=12\theta\frac{\Gamma 2\,\Gamma 3}{\Gamma 5}$$ Therefore $\beta(a,b)=\frac{\Gamma a\,\Gamma b}{\Gamma(a+b)}$

$$E\left[4X_{(1)}\right]=12\theta\frac{(2-1)!(3-1)!}{(5-1)!}$$ Therefore $\Gamma n=(n-1)!$

$$E\left[4X_{(1)}\right]=12\theta\frac{1!\,2!}{4!}=\frac{24\theta}{24}=\theta$$

Hence proved $4X_{(1)}$ is an unbiased estimator of θ

Variance

$$Var\left(4X_{(1)}\right)$$

$$E\left[4X_{(1)}\right]^2=\int_0^\theta \left(4X_{(1)}\right)^2 g(X_{(1)})dX_{(1)}$$

$$E\left[4X_{(1)}\right]^2=\int_0^\theta \left(4X_{(1)}\right)^2 g(X_{(1)})dX_{(1)}$$

$$E\left[4X_{(1)}\right]^2=\int_0^\theta \left(4X_{(1)}\right)^2 \frac{3}{\theta}\left(1-\frac{X_{(1)}}{\theta}\right)^2 dX_{(1)}$$

$$E\left[4X_{(1)}\right]^2=\frac{48}{\theta}\int_0^\theta \left(X_{(1)}\right)^2 \left(1-\frac{X_{(1)}}{\theta}\right)^2 dX_{(1)}$$

Put

$W=\frac{X_{(1)}}{\theta}$ $X_{(1)}\longrightarrow 0$ Then $W\longrightarrow 0$

$W\theta=X_{(1)}$ $X_{(1)}\longrightarrow\theta$ Then $W\longrightarrow 1$

$dX_{(1)}=\theta dW$

$$E\left[4X_{(1)}\right]^2=\frac{8}{5}\theta^2$$

$$E\left[4X_{(1)}\right]^2=48\theta^2\int_0^1 W^{3-1}\left(1-W\right)^{3-1}\,dW$$

Comparing with beta function

a=3 , b= 3

$$E\left[4X_{(1)}\right]^2=48\theta^2\beta\left(3,3\right)$$

$$E\left[4X_{(1)}\right]^2 = 48\theta^2 \frac{\Gamma 3 \Gamma 3}{\Gamma 6} \qquad \beta(a,b) = \frac{\Gamma a \Gamma b}{\Gamma a+b}$$

$$E\left[4X_{(1)}\right]^2 = 48\theta \frac{(3-1)!(3-1)!}{(6-1)!} \qquad \Gamma n = (n-1)!$$

$$E\left[4X_{(1)}\right]^2 = \frac{192}{120}\theta^2$$

Now

$$Var\left(4X_{(1)}\right) = E\left(X_{(1)}\right)^2 - \left[E\left(X_{(1)}\right)\right]^2$$

$$Var\left(4X_{(1)}\right) = \frac{8}{5}\theta^2 - \theta^2$$

$$Var\left(4X_{(1)}\right) = \left(\frac{8-5}{5}\right)\theta^2 = \left(\frac{3}{5}\right)\theta^2$$

ii) $2X_{(2)}^3$

Put i=2 , n=3 , $y_i = x_i$

$$g(y_{(i)}) = \frac{n!}{(\text{i}-1)!\,(\text{n}-\text{i})!}\left[F(y_i)\right]^{i-1}\left[1 - F(y_{i)})\right]^{n-i} f(y_i)$$

$$g(X_{(2)}) = \frac{3!}{0!1!}\left[F(X_{(2)})\right]^{2-1}\left[1 - F(X_{(2)})\right]^{3-2} f(X_{(2)})$$

$$g(X_{(2)}) = 6\left[F(X_{(2)})\right]\left[1 - F(X_{(2)})\right]^1 f(X_{(2)}) \qquad \text{(A)}$$

As

$$f(x) = \frac{1}{\theta} \quad 0 < X < \theta$$

$$F(x) = \int_0^X f(X)d(X) == \int_0^X \frac{1}{\theta}1d(X)$$

$$F(x) = \frac{1}{\theta}X\Big|_0^x$$

$$F(x) = \frac{X}{\theta}$$

$$F(x_2) = \frac{X_{(2)}}{\theta}$$

$$f(x_2) = \frac{1}{\theta}$$

Now Esq. (B) becomes

$$g(X_{(2)}) = 6\left[\frac{X_{(2)}}{\theta}\right]\left[1 - \frac{X_{(2)}}{\theta}\right]\frac{1}{\theta}$$

$$g(X_{(2)}) = \frac{6}{\theta}\left[\frac{X_{(2)}}{\theta}\right]\left[1-\frac{X_{(2)}}{\theta}\right]$$

Now

$$E\left[2X_{(2)}\right] = \int 2X_{(2)} g(X_{(2)}) dX_{(2)}$$

$$E\left[2X_{(2)}\right] = 2\int_0^\theta \frac{6}{\theta}\left(1-\frac{X_{(2)}}{\theta}\right)\left(\frac{X_{(2)}}{\theta}\right) X_{(2)} dX_{(2)} \qquad \text{(B)}$$

Put

$W = \dfrac{X_{(2)}}{\theta}$ $\qquad X_{(2)} \longrightarrow 0$ $\qquad$ Then $W \longrightarrow 0$

$W\theta = X_{(2)}$ $\quad X_{(2)} \longrightarrow \theta$ $\qquad$ Then $W \longrightarrow 1$

$dX_{(2)} = \theta dW$

$$E\left[2X_{(2)}\right] = \frac{12}{\theta}\int_0^1 WW\theta\,(1-W)\theta dW$$

$$E\left[2X_{(2)}\right] = 12\theta\int_0^1 W^{3-1}(1-W)^{2-1} dW \qquad \text{©}$$

As we know that beta function of 1st kind

$$\beta(a,b) = \int_0^1 W^{a-1}(1-w)^{b-1} dW \qquad \text{(D)}$$

Comparing (C) & (D)

a=3 , b= 2

$$E\left[2X_{(2)}\right] = 12\theta\beta(3,2)$$

$$E\left[2X_{(2)}\right] = 12\theta\frac{\Gamma 2\,\Gamma 3}{\Gamma 5} \qquad \text{Therefore } \beta(a,b) = \frac{\Gamma a\,\Gamma b}{\Gamma(a+b)}$$

$$E\left[2X_{(2)}\right] = 12\theta\frac{(2-1)!(3-1)!}{(5-1)!} \qquad \text{Therefore } \Gamma n = (n-1)!$$

$$E\left[2X_{(2)}\right] = 12\theta\frac{1!\,2!}{4!} = \frac{24\theta}{24} = \theta$$

Hence proved $2X_{(2)}$ is an unbiased estimator of θ

Variance

$Var\left(2X_{(2)}\right)$

$$E\left[2X_{(2)}\right]^2 = \int_0^\theta \left(2X_{(2)}\right)^2 g(X_{(2)}) dX_{(2)}$$

$$E\left[2X_{(2)}\right]^2 = \int_0^\theta \left(2X_{(2)}\right)^2 \frac{6}{\theta}\left(\frac{X_{(2)}}{\theta}\right)\left(1-\frac{X_{(2)}}{\theta}\right) dX_{(2)}$$

Put

$$W = \frac{X_{(2)}}{\theta} \qquad X_{(2)} \longrightarrow 0 \qquad \text{Then} \quad W \longrightarrow 0$$

$$W\theta = X_{(2)} \qquad X_{(2)} \longrightarrow \theta \qquad \text{Then} \quad W \longrightarrow 1$$

$$dX_{(2)} = \theta dW$$

$$E\left[2X_{(2)}\right]^2 = \frac{24}{\theta}\int_0^1 (\theta W)^2 (W)(1-W)\theta dW$$

$$E\left[2X_{(2)}\right]^2 = 24\theta^2\int_0^1 (W)^{4-1}(1-W)^{2-1}dW$$

Comparing with beta function

a=3 , b= 2

$$E\left[2X_{(2)}\right]^2 = 24\theta^2\beta(4,2)$$

$$E\left[2X_{(2)}\right]^2 = 24\theta^2\frac{\Gamma 4\,\Gamma 2}{\Gamma 6} \qquad \beta(a,b) = \frac{\Gamma a\,\Gamma b}{\Gamma (a+b)}$$

$$E\left[2X_{(2)}\right]^2 = 24\theta^2\frac{3!}{5!} \qquad \Gamma n = (n-1)!$$

$$E\left[2X_{(2)}\right]^2 = \theta^2\frac{6}{5}$$

Now

$$Var\left(2X_{(2)}\right) = E\left(X_{(2)}\right)^2 - \left[E\left(X_{(2)}\right)\right]^2$$

$$Var\left(2X_{(2)}\right) = \frac{6}{5}\theta^2 - \theta^2$$

$$Var\left(2X_{(2)}\right) = \frac{1}{5}\theta^2$$

iii) $\frac{4}{3}X_{(3)}^3$

Put i=3 , n=3 , $y_i = x_i$

$$g(y_{(i)}) = \frac{n!}{(\text{i}-1)!\,(\text{n}-\text{i})!}\left[F(y_i)\right]^{i-1}\left[1-F(y_{i)})\right]^{n-i} f(y_i)$$

$$g(X_{(3)}) = \frac{3!}{2!\,0!}\left[F(X_{(3)})\right]^{3-1}\left[1-F(X_{(3)})\right]^{3-3} f(X_{(3)})$$

$$g(X_{(3)}) = 3\left[F(X_{(3)})\right]^2 f(X_{(3)}) \qquad \text{(A)}$$

As

$$f(x) = \frac{1}{\theta} \quad 0 < X < \theta$$

$$F(x)=\int_0^X f(X)d(X)==\int_0^X \frac{1}{\theta}1d(X)$$

$$F(x)=\frac{1}{\theta}X\Big|_0^x$$

$$F(x)=\frac{X}{\theta}$$

$$F(x_3)=\frac{X_{(3)}}{\theta}$$

$$f(x_3)=\frac{1}{\theta}$$

Now Esq. (B) becomes

$$g(X_{(3)})=3\left[\frac{X_{(3)}}{\theta}\right]^2\frac{1}{\theta}$$

$$g(X_{(3)})=3\left[\frac{X_{(3)}}{\theta}\right]^2\frac{1}{\theta}$$

Now

$$E\left[\frac{4}{3}X_{(3)}\right]=\frac{4}{3}\int_0^\theta \frac{3}{\theta}\left(\frac{X_{(3)}}{\theta}\right)^2 X_{(3)}dX_{(3)} \qquad \text{(B)}$$

Put

$W=\frac{X_{(3)}}{\theta}$ $\qquad$ $X_{(3)}\longrightarrow 0$ $\qquad$ Then $W\longrightarrow 0$

$W\theta = X_{(3)}$ $\qquad$ $X_{(3)}\longrightarrow \theta$ $\qquad$ Then $W\longrightarrow 1$

$$dX_{(2)}=\theta dW$$

$$E\left[\frac{4}{3}X_{(3)}\right]=\frac{4}{3}\int_0^1 \frac{3}{\theta}(W)^2\theta W\theta dW$$

$$E\left[\frac{4}{3}X_{(3)}\right]=4\theta\int_0^1 W^{4-1}(1-W)^{1-1}dW \qquad \text{©}$$

As we know that beta function of 1st kind

$$\beta(a,b)=\int_0^1 W^{a-1}(1-w)^{b-1}dW \qquad \text{(D)}$$

Comparing (C) & (D)

a=4 , b= 1

$$E\left[\frac{4}{3}X_{(3)}\right]=4\theta\beta(4,1)$$

$$E\left[\frac{4}{3}X_{(3)}\right] = 4\theta\frac{\Gamma 4\,\Gamma 1}{\Gamma 5}$$

$$E\left[\frac{4}{3}X_{(3)}\right] = \theta$$

Therefore $\beta(a,b) = \dfrac{\Gamma a\,\Gamma b}{\Gamma a+b}$

Hence proved $\frac{4}{3}X_{(3)}$ is an unbiased estimator of θ

Variance

$$Var\left(\frac{4}{3}X_{(3)}\right)$$

$$E\left[\frac{4}{3}X_{(3)}\right]^2 = \int_0^\theta \left(\frac{4}{3}X_{(3)}\right)^2 g(X_{(3)})dX_{(3)}$$

$$E\left[\frac{4}{3}X_{(3)}\right]^2 = \int_0^\theta \left(\frac{4}{3}X_{(3)}\right)^2 \frac{3}{\theta}\left(\frac{X_{(3)}}{\theta}\right)^2 dX_{(3)}$$

Put

$W = \dfrac{X_{(3)}}{\theta}$ $\qquad X_{(3)} \longrightarrow 0$ $\qquad$ Then $W \longrightarrow 0$

$W\theta = X_{(3)}$ $\qquad X_{(3)} \longrightarrow \theta$ $\qquad$ Then $W \longrightarrow 1$

$dX_{(3)} = \theta dW$

$$E\left[\frac{4}{3}X_{(3)}\right]^2 = \frac{16}{9}\int_0^1 (\theta W)^2 \frac{3}{\theta}(W)^2 \theta dW$$

$$E\left[\frac{4}{3}X_{(3)}\right]^2 = \frac{16}{3}\theta^2\int_0^1 (W)^{5-1}(1-W)^{1-1}dW$$

Comparing with beta function

a=5 , b= 1

$$E\left[\frac{4}{3}X_{(3)}\right]^2 = \frac{16}{3}\theta^2\beta(5,1)$$

$$E\left[\frac{4}{3}X_{(3)}\right]^2 = \frac{16}{3}\theta^2\frac{\Gamma 5\,\Gamma 1}{\Gamma 6}$$

$$E\left[\frac{4}{3}X_{(3)}\right]^2 = \frac{16}{3}\theta^2\frac{1}{5}$$

$$E\left[\frac{4}{3}X_{(3)}\right]^2 = \theta^2\frac{16}{15}$$

Now

$$Var\left(\frac{4}{3}X_{(3)}\right)=E\left(\frac{4}{3}X_{(3)}\right)^2-\left[E\left(\frac{4}{3}X_{(3)}\right)\right]^2$$

$$Var\left(\frac{4}{3}X_{(3)}\right)=\frac{16}{15}\theta^2-\theta^2$$

$$Var\left(\frac{4}{3}X_{(3)}\right)=\frac{1}{15}\theta^2$$

Q.13:

Let x_1 and x_2 be the tow independent unbiased statistical of the θ. The variance of "x_1" twice of "x_2". If $y=k_1x_1+k_2x_2$ is also unbiased estimator of θ.find the value of k_1 and k_2.For smallest variance of "y".

Solution:

As

$$y=k_1x_1+k_2x_2$$

Applying expectation on b.s

$$E(y)=E(k_1x_1+k_2x_2)$$

$$E(y)=k_1E(x_1)+k_2E(x_2)$$

$$E(y)=k_1\theta+k_2\theta \qquad \therefore E(x_1)=\theta, \quad E(x_2)=\theta$$

$$\theta=\theta(k_1+k_2)$$

$$(k_1+k_2)=1$$

$$(k_2)=1-k_1 \qquad \text{(i)}$$

Now

$$y=k_1x_1+k_2x_2$$

Taking variance on b.s

$$\text{var}(y)=\text{var}(k_1x_1+k_2x_2)$$

$$\text{var}(y)=k_1^2\,\text{var}(x_1)+k_2^2\,\text{var}(x_2) \qquad \therefore \text{var}(x_1)=2v, \quad \text{var}(x_2)=v$$

$$\text{var}(y)=k_1^2\,2v+k_2^2\,v$$

$$\text{var}(y)=k_1^2\,2v+v(1-k_1)^2$$

$$\text{var}(y)=k_1^2\,2v+v(1+k_1^2-2k_1)$$

$$\text{var}(y)=k_1^2\,2v+v+vk_1^2-2vk_1$$

$$\text{var}(y)=k_1^2\,3v+v-2vk_1$$

To minimize the function partially differentiate w.r.t to k_1 and equating to zero.

$$\frac{\partial v(y)}{\partial k_1} = \frac{\partial}{\partial k_1}(k_1{}^2 3v + v - 2vk_1)$$

$$0 = 6vk_1 + 0 - 2v$$

$$0 = 2v(3k_1 - 1)$$

$$0 = (3k_1 - 1)$$

$3k_1 = 1$ $\qquad$ $k_1 = \frac{1}{3}$

$k_2 = 1 - k_1$

$$k_2 = 1 - \frac{1}{3}$$

$$k_2 = \frac{2}{3}$$

Q.14:

If s^2 is an unbiased estimator of σ^2 then show that ‘s’ is a biased estimator of ‘ σ ‘.

Solution:

It is given that

$$E(s^2) = \sigma^2$$

$$Var(s) \geq 0$$

$$E(s)^2 - (E(s))^2 \geq 0$$

$$E(s)^2 \geq (E(s))^2$$

$$\sigma^2 \geq (E(s))^2$$

$$\sigma \geq E(s)$$

Hence Proved that ‘s’ is a biased estimator of ‘ σ ‘.

Q.15:

If $X_1, X_2, ..., X_n$ be a random a sample of normal population with mean’ u’ and variance’ 1’. Then show that $T = \frac{\sum X_i^2}{n}$ is an unbiased estimator of $\mu^2 + 1$.

Solution:

$$T = \frac{\sum X_i^2}{n}$$

Applying expectation on both sides

$$E(T) = E\left(\frac{\sum X_i^2}{n}\right)$$

$$E(T) = \frac{\sum E(X_i)^2}{n}$$

$$E(T) = \frac{n}{n} E(X_i)^2$$

$$E(T) = E(X_i)^2$$

$\text{var}(X) = E(X^2) - (E(X))^2$ Therefore $E(X) = \mu$ $Var(X) = 1$

$$1 = E(X^2) - \mu^2$$

$$E(X^2) = 1 + \mu^2$$

$$E(T) = 1 + \mu^2$$

Hence $T = \frac{\sum X_i^2}{n}$ is an unbiased estimator of $1 + \mu^2$.

Q.16:

A random variable "x" is a uniformly distribution over the range $0<x<\alpha$ where "α" is the unknown parameter of sample of "n" independent observaion are arranged in increasing order of magnitude as $x_1^n < x_2 \ldots\ldots\ldots\ldots x_n^n$.

Drive sampling distribution of x_r^n the r^{th} order statistic and verify that.

$$E(X_r) = \frac{\alpha r}{(n+1)} \text{ and } Var(X_r) = \frac{\alpha^2 r}{(n+1)}\left[\frac{n+1-r}{(n+2)}\right]$$

a) As shown that $X_{(n)}$ is a biased estimator of α. But bias tends to zero as $n \sim \infty$.
b) for "n" is odd both the sample mean and sample median are un baised estimator of $\frac{\alpha}{2}$ and the ratio of their variance $\frac{n+2}{3}$

Solution:

As we know that pdf of rth order statistics

$$g(X_r) = \frac{n![F(X_r)]^{r-1}[1-F(X_r)]^{n-r}}{(r-1)!(n-r)!} f(X_r)$$

$$f(X) = \frac{1}{\alpha} \quad 0 < X < \infty$$

$$f(X_r) = \frac{1}{\alpha} \quad 0 < X_r < \infty$$

$$F(X) = \int_0^X f(x)dx$$

$$F(X) = \int_0^X f(x)dx$$

$$F(X) = \int_0^X \frac{1}{\alpha} dx = \frac{X}{\alpha}$$

$$F(X_r) = \frac{X_r}{\alpha}$$

$$g(X_r) = \frac{n!\left[\frac{X_r}{\alpha}\right]^{r-1}\left[1-\frac{X_r}{\alpha}\right]^{n-r}}{(r-1)!(n-r)!}\frac{1}{\alpha}$$

$$E(X_r) = \int_0^{\infty} X_r g(X_r)dx$$

$$E(X_r) = \int_0^{\alpha} X_r \frac{n!\left[\frac{X_r}{\alpha}\right]^{r-1}\left[1-\frac{X_r}{\alpha}\right]^{n-r}}{(r-1)!(n-r)!}\frac{1}{\alpha}dx$$

$$E(X_r) = \frac{n!}{(r-1)!(n-r)!}\int_0^{\alpha} X_r \left[\frac{X_r}{\alpha}\right]^{r-1}\left[1-\frac{X_r}{\alpha}\right]^{n-r}\frac{1}{\alpha}dx$$

Let $W = \frac{X_r}{\alpha}$ Limits

$X_r = \alpha W$ $X_r \to \alpha \; Then \; W \to 1$

$dX_r = \alpha dW$ $X_r \to 0 \; Then \; W \to 0$

$$E(X_r) = \frac{n!}{(r-1)!(n-r)!}\int_0^{1} \alpha W [W]^{r-1}[1-W]^{n-r}\frac{1}{\alpha}\alpha dW$$

$$E(X_r) = \frac{\alpha n!}{(r-1)!(n-r)!}\int_0^{1} W^{r+1-1}[1-W]^{n-r+1-1}dW$$

Comparing with beta function of 1st kind

$$\beta_1(a,b) = \int_0^1 W^{a-1}[1-W]^{b-1}dW$$

a=r+1 b=n-r+1

$$E(X_r) = \frac{\alpha n!}{(r-1)!(n-r)!}\beta_1(r+1, n-r+1)$$

$$E(X_r) = \frac{\alpha n! r!(n-r)!}{(r-1)!(n-r)!(n+1)!}$$

$$E(X_r) = \frac{\alpha n! r!}{(r-1)!(n+1)!}$$

$$E(X_r) = \frac{\alpha n! r(r-1)!}{(r-1)!(n+1)n!}$$

$$E(X_r) = \frac{\alpha r}{(n+1)}$$

$$E(X_r)^2 = \int_0^{\infty} X^2{}_r g(X_r)dx$$

$$E(X_r^2) = \int_0^\alpha X_r^2 \frac{n!\left[\frac{X_r}{\alpha}\right]^{r-1}\left[1-\frac{X_r}{\alpha}\right]^{n-r}}{(r-1)!(n-r)!}\frac{1}{\alpha}dx$$

$$E(X_r^2) = \frac{n!}{(r-1)!(n-r)!}\int_0^\alpha X_r^2\left[\frac{X_r}{\alpha}\right]^{r-1}\left[1-\frac{X_r}{\alpha}\right]^{n-r}\frac{1}{\alpha}dx$$

Let $W = \frac{X_r}{\alpha}$ Limits

$X_r = \alpha W$ $X_r \to \alpha \; Then \; W \to 1$

$dX_r = \alpha dW$ $X_r \to 0 \; Then \; W \to 0$

$$E(X_r^2) = \frac{n!}{(r-1)!(n-r)!}\int_0^1 (\alpha W)^2 [W]^{r-1}[1-W]^{n-r}\frac{1}{\alpha}\alpha dW$$

$$E(X_r^2) = \frac{\alpha^2 n!}{(r-1)!(n-r)!}\int_0^1 W^{r+2-1}[1-W]^{n-r+1-1}dW$$

Comparing with beta function of 1st kind

$$\beta_1(a,b) = \int_0^1 W^{a-1}[1-W]^{b-1}dW$$

a=r+2 b=n-r+1

$$E(X_r^2) = \frac{\alpha^2 n!}{(r-1)!(n-r)!}\beta_1(r+2, n-r+1)$$

$$E(X_r^2) = \frac{\alpha^2 n!(r+1)!(n-r)!}{(r-1)!(n-r)!(n+2)!}$$

$$E(X_r^2) = \frac{\alpha^2 n!(r+1)r(r-1)!}{(r-1)!(n+2)(n+1)n!}$$

$$E(X_r^2) = \frac{\alpha^2 (r+1)r}{(n+2)(n+1)}$$

$$Var(X_r) = E(X_r^2) - \left[E(X_r^2)\right]^2 = \frac{\alpha^2(r+1)r}{(n+2)(n+1)} - \left[\frac{\alpha r}{(n+1)}\right]^2$$

$$Var(X_r) = \frac{\alpha^2 r}{(n+1)}\left[\frac{(r+1)}{(n+2)} - \frac{r}{(n+1)}\right]$$

$$Var(X_r) = \frac{\alpha^2 r}{(n+1)}\left[\frac{(r+1)(n+1) - r(n+2)}{(n+2)}\right]$$

$$Var(X_r) = \frac{\alpha^2 r}{(n+1)}\left[\frac{nr + r + n + 1 - rn - 2r}{(n+2)}\right]$$

$$Var(X_r) = \frac{\alpha^2 r}{(n+1)}\left[\frac{n+1-r}{(n+2)}\right]$$

$E(x_n) = \dfrac{n\alpha}{n+1} \neq \alpha$

Hence $X_{(n)}$ is a biased estimator of α.

$E(X_n) = \dfrac{\alpha(n+1-1)}{n+1} = \dfrac{\alpha(n+1)}{n+1} - \dfrac{\alpha}{n+1}$

As n $\longrightarrow \infty then$

$E(X_n) = \alpha - \text{limt}_{n\to\infty} \dfrac{\alpha}{n+1}$

$E(X_n) = \alpha - 0$

$E(X_n) = \alpha$

Hence proved When n $\to \infty$ then baised $\to$ 0

ii)
As we know that

$E(X_r) = \dfrac{\alpha r}{n+1}$

If 'n' is odd no we may replace n=2v+1 where v=0,1,2………..

$\text{Median} = \dfrac{(n+1)}{2} th\ \ value$

$\text{Median} = \dfrac{(2v+1+1)}{2} th\ \ value$

$\text{Median} = (v+1) th\ \ value$

So,

$E(X_{med}) = \dfrac{\alpha(v+1)}{2v+1+1}$

$E(X_{med}) = \dfrac{\alpha}{2}$

Now

$E(\overline{X}) = E(\dfrac{\sum X}{n}) = \dfrac{n}{n} E(X) = E(X)$

$E(X) = \int x f(x) dx$

$E(X) = \dfrac{1}{\alpha} \int_0^{\alpha} x dx$

$E(X) = \dfrac{1}{\alpha} \dfrac{x^2}{2} \Big|_0^{\alpha}$

$E(X) = \dfrac{1}{\alpha} \dfrac{\alpha^2}{2}$

$E(X) = \dfrac{\alpha}{2}$

$E(\overline{x}) = \dfrac{\alpha}{2}$

Hence sampling distribution of Median and sampling of mean both are unbiased estimators of $\dfrac{\alpha}{2}$

$$\text{Var }(X_r) = \frac{\alpha^2 r(n-r+1)}{(n+1)^2(n+2)}$$

Put r=v+1 & n=2v+1

$$\text{Var }(X_{med}) = \frac{\alpha^2(v+1)(2v+1-v-1+1)}{(2v+1+1)^2(2v+1+2)}$$

$$\text{Var }(X_{med}) = \frac{\alpha^2(v+1)(v+1)}{4(v+1)^2(2v+1+2)}$$

$$\text{Var }(X_{med}) = \frac{\alpha^2}{4(2v+3)}$$

$$\text{Var }(X_{med}) = \frac{\alpha^2}{4(n+2)} \qquad \therefore n = 2v+1$$

$$\text{Var }(X_{med}) = \frac{\alpha^2}{4(n+2)}$$

And

$$Var(\bar{x}) = Var(\frac{\sum x}{n}) = \frac{n}{n^2}v(x) = \frac{1}{n}v(x)$$

$$E(X^2) = \int x^2 f(X)dX$$

$$\text{E }(X^2) = \frac{1}{\alpha}\frac{\alpha^3}{3}$$

$$E(X^2) = \frac{\alpha^2}{3}$$

Var(x)=E(x^2)-(E(x))^2

$$Var(X) = \frac{\alpha^2}{3} - (\frac{\alpha}{2})^2$$

$$Var(X) = \frac{\alpha^2}{3} - (\frac{\alpha}{2})^2$$

$$Var(X) = \frac{\alpha^2}{12}$$

$$Var(\bar{X}) = \frac{\alpha^2}{12n}$$

And

$$\text{Ratio } = \frac{v(\bar{x})}{v(x_{med})} = \frac{\frac{\alpha^2}{12n}}{\frac{\alpha^2}{4(n+2)}}$$

$$Ratio = \frac{4(n+2)}{12n} = \frac{n+2}{3n}$$

Proved

Q.17:

Show that (estimator) in a normal distribution with mean zero and standard deviation σ then

$$T_1 = \frac{\left[\frac{1}{2}\sum X_i^2\right]^{\frac{1}{2}}\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}} \text{ and } T_2 = \frac{\left[\frac{1}{2}\sum (X-\bar{X})^2\right]^{\frac{1}{2}}\Gamma\frac{n-1}{2}}{\Gamma\frac{n}{2}}$$ both are unbiased estimator of σ.

Solution:

As

$$T_1 = \frac{\left[\frac{1}{2}\sum X_i^2\right]^{\frac{1}{2}}\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}}$$

Taking expectation on both sides

$$E(T_1) = E\left[\frac{\left[\frac{1}{2}\sum X_i^2\right]^{\frac{1}{2}}\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}}\right]$$

$$E(T_1) = \left[\frac{\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}}\right]E\left[\frac{1}{2}\sum X_i^2\right]^{\frac{1}{2}}$$

$$E(T_1) = \left[\frac{\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}}\right]\left[\frac{1}{2}\right]^{\frac{1}{2}}E\left[\sum X_i^2\right]^{\frac{1}{2}}$$

As we know that

$$\chi_n^2 = \sum_{i=1}^{n}\frac{(X_i-\mu)^2}{\sigma^2}$$

$$\sigma^2\chi_n^2 = \sum_{i=1}^{n}(X_i-\mu)^2$$

$$\sigma^2\chi_n^2 = \sum_{i=1}^{n}(X_i)^2 \qquad\qquad \mu=0$$

$$E\left[\sum X_i^2\right]^{\frac{1}{2}} = E\left[\sigma^2\chi_n^2\right]^{\frac{1}{2}} = \sigma E\left[\chi_n^2\right]^{\frac{1}{2}}$$

$$E\left[\sum X_i^2\right]^{\frac{1}{2}} = \sigma E\left[\chi_n^2\right]^{\frac{1}{2}} = \int_0^\infty\left[\chi_n^2\right]^{\frac{1}{2}} f(\chi_n^2)d\chi_n^2$$

$$f(\chi_n^2) = \frac{1}{\Gamma\frac{n}{2}2^{\frac{n}{2}}}\left(\chi_n^2\right)^{\frac{n}{2}-1}e^{-\frac{\chi_n^2}{2}}$$

$$E\left[\sum X_i^2\right]^{\frac{1}{2}} = \sigma E\left[\chi_n^2\right]^{\frac{1}{2}} = \sigma\int_0^\infty\left[\chi_n^2\right]^{\frac{1}{2}}\frac{1}{\Gamma\frac{n}{2}2^{\frac{n}{2}}}\left(\chi_n^2\right)^{\frac{n}{2}-1}e^{-\frac{\chi_n^2}{2}}d\chi_n^2$$

$$E\left[\sum X_i^2\right]^{\frac{1}{2}} = \sigma E\left[\chi_n^2\right]^{\frac{1}{2}} = \sigma \frac{1}{\Gamma\frac{n}{2}\, 2^{\frac{n}{2}}} \int_0^\infty \left[\chi_n^2\right]^{\frac{1}{2}} \left(\chi_n^2\right)^{\frac{n}{2}-1} e^{-\frac{\chi_n^2}{2}} d\chi_n^2$$

$$E\left[\sum X_i^2\right]^{\frac{1}{2}} = \sigma E\left[\chi_n^2\right]^{\frac{1}{2}} = \sigma \frac{1}{\Gamma\frac{n}{2}\, 2^{\frac{n}{2}}} \int_0^\infty \left(\chi_n^2\right)^{\frac{n+1}{2}-1} e^{-\frac{\chi_n^2}{2}} d\chi_n^2$$

Comparing with gamma function and we get

$$E\left[\sum X_i^2\right]^{\frac{1}{2}} = E\left[\chi_n^2\right]^{\frac{1}{2}} = \sigma \frac{1}{\Gamma\frac{n}{2}\, 2^{\frac{n}{2}}} \Gamma\frac{n+1}{2}\, 2^{\frac{n+1}{2}} = \sigma \frac{1}{\Gamma\frac{n}{2}} \Gamma\frac{n+1}{2}\, 2^{\frac{1}{2}}$$

$$E(T_1) = \left[\frac{\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}}\right] \sigma \frac{1}{\Gamma\frac{n}{2}} \Gamma\frac{n+1}{2} = \sigma$$

Hence T_1 is an unbiased estimator of σ

Similarly

$$T_2 = \frac{\left[\frac{1}{2}\sum (X-\overline{X})^2\right]^{\frac{1}{2}} \Gamma\frac{n-1}{2}}{\Gamma\frac{n}{2}}$$

Taking expectation on both sides

$$E(T_2) = E\left[\frac{\left[\frac{1}{2}\sum (X-\overline{X})^2\right]^{\frac{1}{2}} \Gamma\frac{n-1}{2}}{\Gamma\frac{n}{2}}\right]$$

$$E(T_2) = \frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n}{2}} E\left[\frac{1}{2}\sum (X-\overline{X})^2\right]^{\frac{1}{2}}$$

$$E(T_2) = \frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n}{2}} \left[\frac{1}{2}\right]^{\frac{1}{2}} E\left[\sum (X-\overline{X})^2\right]^{\frac{1}{2}}$$

As we know that

$$\chi^2_{n-1} = \sum_{i=1}^{n} \frac{\left(X_i - \overline{X}\right)^2}{\sigma^2}$$

$$\sigma^2 \chi^2_{n-1} = \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$$

$$E\left[\sum (X - \overline{X})^2\right]^{\frac{1}{2}} = E(\sigma^2 \chi^2_{n-1})^{\frac{1}{2}} = \sigma E\left[\chi^2_{n-1}\right]^{\frac{1}{2}}$$

$$E\left[\sum (X - \overline{X})^2\right]^{\frac{1}{2}} = \sigma E\left[\chi^2_{n-1}\right]^{\frac{1}{2}} = \int_0^{\infty} \left[\chi^2_{n-1}\right]^{\frac{1}{2}} f(\chi^2_{n-1}) d\chi^2_{n-1}$$

$$f(\chi^2_{n-1}) = \frac{1}{\left\lceil\frac{n-1}{2} 2^{\frac{n-1}{2}}\right.} \left(\chi^2_{n-1}\right)^{\frac{n-1}{2}-1} e^{-\frac{\chi^2_{n-1}}{2}}$$

$$E\left[\sum (X - \overline{X})^2\right]^{\frac{1}{2}} = \sigma E\left[\chi^2_{n-1}\right]^{\frac{1}{2}} = \sigma \int_0^{\infty} \left[\chi^2_{n-1}\right]^{\frac{1}{2}} \frac{1}{\left\lceil\frac{n-1}{2} 2^{\frac{n-1}{2}}\right.} \left(\chi^2_{n-1}\right)^{\frac{n-1}{2}-1} e^{-\frac{\chi^2_{n-1}}{2}} d\chi^2_{n-1}$$

$$E\left[\sum (X - \overline{X})^2\right]^{\frac{1}{2}} = \sigma E\left[\chi^2_{n-1}\right]^{\frac{1}{2}} = \frac{1}{\left\lceil\frac{n-1}{2} 2^{\frac{n-1}{2}}\right.} \sigma \int_0^{\infty} \left(\chi^2_{n-1}\right)^{\frac{n-1+1}{2}-1} e^{-\frac{\chi^2_{n-1}}{2}} d\chi^2_{n-1}$$

$$E\left[\sum (X - \overline{X})^2\right]^{\frac{1}{2}} = \sigma E\left[\chi^2_{n-1}\right]^{\frac{1}{2}} = \frac{1}{\left\lceil\frac{n-1}{2} 2^{\frac{n-1}{2}}\right.} \sigma \int_0^{\infty} \left(\chi^2_{n-1}\right)^{\frac{n}{2}-1} e^{-\frac{\chi^2_{n-1}}{2}} d\chi^2_{n-1}$$

Comparing with gamma function and we get

$$E\left[\sum (X - \overline{X})^2\right]^{\frac{1}{2}} = E\left[\chi^2_n\right]^{\frac{1}{2}} = \sigma \frac{1}{\left\lceil\frac{n-1}{2} 2^{\frac{n-1}{2}}\right.} \left\rceil\frac{n}{2} 2^{\frac{n}{2}}\right.$$

$$E(T_2) = \frac{\left\lceil\frac{n-1}{2}\right.}{\left\lceil\frac{n}{2}\right.} \left[\frac{1}{2}\right]^{\frac{1}{2}} \sigma \frac{1}{\left\lceil\frac{n-1}{2} 2^{\frac{n-1}{2}}\right.} \left\rceil\frac{n}{2} 2^{\frac{n}{2}}\right.$$

$$E(T_2) = \sigma$$

Hence T_2 is an unbiased estimator of σ

Q.18

If X1, X2,……. , Xn from a population with mean "M" and variance "σ^2" and the correlation coefficient between any two observation is ρ. If $T = h\sum X_i^2 + k(\sum X_i)^2$ where 'h' and 'k' are unknown constant and 'T' is an unbiased estimate of "σ^2". Then show that

$$T = \frac{\sum (X - \overline{X})^2}{(1-\rho)(n-1)}$$

Solution:

Let

$$T = h\sum X_i^2 + k(\sum X_i)^2 \qquad \text{(A)}$$

Applying expectation on both sides

$$E[T] = E[h\sum X_i^2 + k(\sum X_i)^2]$$

$$E[T] = h[E(\sum X_i^2) + kE(\sum X_i)^2]$$

$$E[T] = h\sum^n E(X_i^2) + kE\left[\sum^n X_i^2 + \sum_{i\neq j}^{n(n-1)}\sum_{i\neq j}^{n(n-1)} X_i X_j\right]$$

$$E[T] = nhE(X_i^{\ 2}) + k\left[nE(X_i^2) + n(n-1)E(X_i X_j)\right]$$

$$E[T] = nhE(X_i^{\ 2}) + \left[nkE(X_i^2) + nk(n-1)E(X_i X_j)\right] \qquad \text{(B)}$$

Side note:

V(x_i)= E(x^2)-(E(x))^2

$E(x_i) = M \quad E(x_j) = M$

$\delta^2 = E(x^2) - M^2$

$E(x^2) = \delta^2 + M^2$

Cov (x_i , x_j) =E(x_i ,x_j)-E(x_i) E(x_j)

Cov (x_i , x_j) =E(x_i ,x_j)- M^2

$\rho = \dfrac{\text{cov}(x_i, x_j)}{\sqrt{v(x_i), v(x_j)}}$ v (x_i)=v(x_j)

$\rho = \dfrac{\text{cov}(x_i, x_j)}{\delta^2}$

Cov(x_i ,x_j)=$\delta^2\rho$

$\delta^2\rho$ =E(x_i ,x_j) - M^2

$E(X_i X_j) = M^2 + \delta^2\rho$

Put in equation (B)

$$E(T) = nh(\delta^2 + M^2) + nk(\delta^2 + M^2) + kn(n-1)(\delta^2\rho + M^2)$$

$$E(T) = nh\delta^2 + nhM^2 + nk\delta^2 + nkM^2 + kn(n-1)\delta^2\rho + nk(n-1)M^2$$
$$E(T) = nh\delta^2 + nhM^2 + nk\delta^2 + nkM^2 + kn^2\delta^2\rho - nk\delta^2\rho + n^2kM^2 - nkM^2$$

$$E(T) = (nh\delta^2 + nk\delta^2 + kn^2\delta^2\rho - nk\delta^2\rho) + n^2kM^2 + nhM^2$$

As ‘T’ is an unbiased estimator of σ^2 if and only if

$n^2kM^2 + nhM^2 = 0$ (i)

$\delta^2(nh + nk + kn^2\rho - nk\rho) = \delta^2$

$nh + nk + kn^2\rho - nk\rho = 1$ (ii)

Now

$n^2kM^2 + nhM^2 = 0$

$nM^2(nk + h) = 0$

$nk + h = 0$

$h = -nk$

$n(-nk) + nk + kn^2\rho - nk\rho = 1$

$-n^2k + nk + kn^2\rho - nk\rho = 1$

$nk(-n + 1 - n\rho - \rho) = 1$

$nk[-1(n-1) + \rho(n-1)] = 1$

$nk[(\rho - 1)(n-1)] = 1$

$nk = {}^{1}\!/_{(n-1)(\rho-1)}$

Now

$h = -nk$

$h = -\left[{}^{1}\!/_{(n-1)(1-\rho)}\right]$

$h = {}^{-1}\!/_{-(n-1)(1-\rho)}$

$h = {}^{1}\!/_{(n-1)(1-\rho)}$

Again

$h = -nk$

${}^{1}\!/_{(n-1)(1-\rho)} = -nk$

${}^{-1}\!/_{n(n-1)(1-\rho)} = k$

$k = {}^{-1}\!/_{n(n-1)(1-\rho)}$

Now

$T = h\sum X_i^2 + k(\sum X_i)^2$

$$T = \sum X_i^2 \frac{1}{(n-1)(1-\rho)} + \frac{-1}{n(n-1)(1-\rho)}(\sum X_i)^2$$

$$T = \frac{1}{(n-1)(1-\rho)} \sum (X_i - \overline{X})$$

As required result

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